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Derivation of Poincaré Covariance from Causality Requirements in Field Theory†

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Abstract

It is proved that the group of covariance of a non-second quantized theory of scalar fields on Minkowski space is uniquely restricted to the causality group, constituted by the group of dilatations and by the orthochronous Poincaré group, if certain causality requirements in field theory are assumed.

1. *Introduction*

E. C. Zeeman (1964) proved an interesting theorem stating that a causal relation among points of a Minkowski space implies Poincaré covariance.

More explicitly, Zeeman considers a $(3+1)$-dimensional Minkowski space M with metric tensor

$$Q(x,x') = (x^0 - x'^0)^2 - (\mathbf{x} - \mathbf{x}')^2 \tag{1.1}$$

equipped with a partial ordering

$$x < x' \tag{1.2}$$

which states that an event at a point $x \in M$ is causally related to an event at a point $x' \in M$, when the separation is time-like

$$Q(x,x') > 0 \tag{1.3}$$

and there is an ordering in time, e.g.

$$x^0 < x'^0 \tag{1.4}$$

Then, the largest group of automorphisms of M preserving relation (1.2) is uniquely restricted to the so-called *causality group C* which is constituted by the following groups of transformations in M:

(1) group of dilatations D, i.e., the multiplicative group of (positive) real numbers;
(2) group of homogeneous orthochronous Lorentz transformations $L\uparrow$;
(3) group of translations T.

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It is the purpose of the present paper to show that an equivalent result occurs also for a non-second quantized field theory in Minkowski space. More explicitly we shall see that, if one associates fields $\psi(x)$ to points $x \in M$, certain causality requirements of field theories imply Poincaré covariance.

In Section 2 we recall some basic facts and causal features of a scalar field theory covariant under the Poincaré group; in Section 3 we introduce a scalar field theory covariant under an arbitrary group of automorphisms G of the Minkowski space, preserving the causal behavior; in Section 4 we prove that the largest group of automorphisms G is then uniquely restricted to the causality group C; finally, in Section 5 we introduce some supplementary remarks and comments.

2. *Causal Features of a Poincaré Covariant Scalar Field Theory*

Let us recall some basic facts of a theory of scalar fields $\varphi(x)$ as the Hilbert space H_φ of all smooth solutions of the Klein–Gordon equation

$$(\Box + m^2)\varphi(x) = 0, \qquad m > 0 \tag{2.1}$$

As is well known, fields $\varphi(x)$ can be decomposed into the independent sets of positive and negative frequency solutions

$$\varphi^{\pm}(x) = (2\pi)^{-2} \int dp \exp(\mp ipx)\, \theta(\pm p^0)\, \delta(p^2 - m^2)\tilde{\varphi}^{\pm}(\mathbf{p}) \tag{2.2}$$

where $\tilde{\varphi}^{\pm}(\mathbf{p})$ are infinitely continuously differentiable functions with compact support.

Under a unitary irreducible representation $R_{(\Lambda,a)}$ of the restricted Poincaré group $P_+^\uparrow$ in H_φ, the fields $\varphi(x)$ transform covariantly according to

$$R_{(\Lambda,a)}\varphi(x) = \varphi[(\Lambda, a)x] \tag{2.3}$$

The local propagator $\Delta(x)$ of the theory is a solution of the equation

$$(\Box + m^2)\Delta(x) = 0 \tag{2.4}$$

and is uniquely determined by the conditions

$$\Delta(0, \mathbf{x}) = 0, \qquad \left(\frac{\partial \Delta(x)}{\partial x^0}\right)_{x^0=0} = -\delta(\mathbf{x}) \tag{2.5}$$

$\Delta(x)$ enables us to solve the Cauchy or initial value problem of equation (2.1). Indeed, if the field $\varphi(x)$, together with its first time derivative $\dot{\varphi}(x)$, is known, then the field at arbitrary space-time points $x' \in M$ is given by

$$\varphi(x') = \int [\Delta(x' - x)\dot{\varphi}(x) - \dot{\Delta}(x' - x)\varphi(x)]\, d\mathbf{x} \tag{2.6}$$

The above relation is usually generalized to an explicit Poincaré covariant form by introducing a space-like surface σ_0 in M with unit normal $n_\mu(x)$

($\mu = 0, 1, 2, 3$). Then, once $\varphi(x)$ and $n_\mu(x)\partial^\mu\varphi(x)$ are known on σ_0, the values of $\varphi(x)$ at any later space-like surface is given by

$$\varphi(x') = \int \{\Delta(x'-x)\, n_\mu \partial^\mu \varphi(x) - [n_\mu \partial^\mu \Delta(x'-x)]\,\varphi(x)\}\, d\sigma(x) \quad (2.7)$$

The field $\varphi(x')$ constructed according to (2.6) or (2.7) preserves all the essential features and transformation properties of $\varphi(x)$, and it will be called in the following the 'propagated field' of $\varphi(x)$.

Let us now recall the following causal features satisfied by the above field theory.

(I) *Localizability Condition*

This condition requires that the region where the solutions (2.1) do not vanish must be localized in space-time.

The above condition can be expressed by assuming that the relation (Chandler, 1968, 1969)

$$\lim_{\tau=\infty} \tau^n \varphi(\hat{x}\tau) = 0 \quad (2.8)$$

is verified for every positive integer n and uniformly for any $\hat{x}$ on the complement of the domain

$$V(\tilde{\varphi}) = (\hat{x}|\hat{x} = \kappa t;\ \kappa \epsilon\, \mathrm{supp}\,\tilde{\varphi},\ t \text{ real}) \quad (2.9)$$

This implies the existence of positive numbers ϵ, δ, τ_0 and a positive integer n such that (Chandler, 1968, 1969)

$$|\varphi(\hat{x}\tau)| < \frac{\delta}{(|\hat{x}|\tau)^n} \quad (2.10)$$

for all $\tau > \tau_0$ and $\hat{x}$ in the complement of the cone with vertex at the origin

$$V_\epsilon(\varphi) = (\hat{x}|\,|\hat{x}| \leqslant \epsilon;\ \text{or } |\hat{x}t^{-1} - \kappa| \leqslant \epsilon \text{ for some } \kappa \epsilon\, \mathrm{supp}\,\varphi;\ t \text{ real} \neq 0) \quad (2.11)$$

where $|x|$ is the Euclidean distance

$$\left(\sum_0^3{}_\mu\, x_\mu{}^2\right)^{1/2}$$

Then the fields rapidly and uniformly collapse on $V_\epsilon(\varphi)$ as $\tau \to \infty$ and we can say that $\varphi(x)$ is confined to the region

$$V_\epsilon(\varphi; \tau) = (x|x = \hat{x}\tau;\ \hat{x} \in V_\epsilon(\varphi),\ \tau \text{ real}) \quad (2.12)$$

Localizability condition is one of the basic requirements of any causal field theory, since it allows the same distinction between ingoing and outgoing waves. In our case, conditions (2.8) or (2.10) are satisfied by solutions of the form (2.2) (Chandler, 1968, 1969).

(II) *Causal Way of Propagation*

This condition requires that if a field $\varphi(x)$, together with its first time derivative, is confined to a finite region of space at a given time, then the propagated field $\varphi(x')$ must be localized too in a finite region of space (Helgewood, 1962).

In order to clarify this requirement, let us consider only positive frequency solutions $\varphi^+(x)$ of (2.1). Then (2.6) becomes

$$\varphi^+(x') = \int [\Delta^+(x'-x)\dot{\varphi}^+(x) - \dot{\Delta}^+(x'-x)\varphi(x)]\,d\mathbf{x} \tag{2.13}$$

where, as usual, $\Delta^+(x)$ comes from the decomposition

$$\Delta(x) = \Delta^+(x) + \Delta^-(x) \tag{2.14}$$

As an example of solutions of (2.1) which is completely localized at the origin at the time $x^0 = 0$ we assume

$$\varphi(0, \mathbf{x}) = \delta(\mathbf{x}), \qquad \dot{\varphi}(0, \mathbf{x}) = 0 \tag{2.15}$$

Then, by substituting in (2.14) we have

$$\varphi(x') = -\left(\frac{\partial \Delta^+(x'-x)}{\partial x^0}\right)_{x^0=0,\,\mathbf{x}=0} \tag{2.16}$$

which shows that the propagated field $\varphi(x')$ does not longer vanish outside of a finite region of space on account of the fact that $\Delta^+(x)$ does not satisfy properties (2.5) of $\Delta(x)$.

The formulae for propagated fields are also interpreted as expressing the way according to which fields develop in the course of time (Helgewood, 1962). Then field (2.15), propagated according to (2.16), is said to spread with infinite velocity, while a causal way of propagation requires that the velocity of propagation of the fronts must be bounded by the velocity of light.

Thus, a scalar field theory constituted of only positive frequency solutions (or equivalently of only negative frequency solutions) of (2.1) possesses an acausal way of propagation (Hilgewoord, 1962). On the contrary, the full solutions $\varphi(x)$, propagated according to (2.6) or (2.7) possess a causal way of propagation on account of properties (2.5) of the propagator $\Delta(x)$.

(III) *Time Ordering*

A succession of events at points $x, x', x'', \ldots EM$ can be assumed either with time ordering $x^0 < x^{0'} < x^{0''} < \ldots$ or with the inverse ordering $x^0 > x^{0'} > x^{0''} > \ldots$. Causal theories can be constructed either with a given ordering in time or with its inverse. However, the same definition of cause and effect requires the assumption of a fixed ordering in time. In field theories, this assumption allows the very definition of ingoing and outgoing waves.

3. *Causal Field Theory Covariant Under an Arbitrary Group*

The purpose of this section is to introduce a scalar field theory covariant under an arbitrary group, but preserving the causal features pointed out in the preceding section.

Let us introduce first the following *basic assumptions*:

(a) We work in a (3 + 1)-dimensional Minkowski space M with metric tensor (1.1), equipped with a group of automorphisms G.

(b) Let H_ψ be the Hilbert space of smooth (in the meaning of Section 2) 'fields' $\psi(x)$, with $x \in M$, solutions of a linear second-order differential equation

$$(D^{(2)} + \mu^2)\psi(x) = 0, \qquad \mu > 0 \tag{3.1}$$

and transforming covariantly under a unitary representation T_g of G in H_ψ for any $g \in G$, according to

$$T_g \psi(x) = \psi(gx) \tag{3.2}$$

(c) Corresponding to any field $\psi(x) \in H_\psi$, we assume the existence of a propagated field $\psi(x') \in H_\psi$, with $x' \in M$, preserving all the essential features and transformations properties of $\psi(x)$.

We realize requirement (c) by assuming that, as a Cauchy value problem of (3.1), when a field $\psi(x)$ and its first time derivative $\dot{\psi}(x)$ are known, then the field at an arbitrary point $x' \in M$ is given by a kernel dependent relation that we write symbolically

$$\psi(x') = \int F[k(x', x), \psi(x), \dot{\psi}(x)]\, d\mathbf{x} \tag{3.3}$$

Let us note that assumptions (a), (b) and (c) do not imply Poincaré covariance and our field theory is largely arbitrary on account of the arbitrariness of the group of automorphisms G of M.

Now we restrict our field theory with the following causality conditions.

(I) *Localizability Condition*

As pointed out in Section 2, we assume that the fields $\psi(x)$ vanish outside of a localized region in space-time, and we express this condition by requiring that for every positive integer n

$$\lim_{\tau=\infty} \tau^n \psi(\hat{x}\tau) = 0 \tag{3.4}$$

uniformly for any $\hat{x}$ in the complement of the domain

$$V(\tilde{\psi}) = (\hat{x} | \hat{x} = \kappa t,\ \kappa \in \operatorname{supp} \tilde{\psi},\ t \text{ real}) \tag{3.5}$$

where $\tilde{\psi}$ is now the Fourier transform of ψ, or, equivalently, by assuming the existence of positive numbers ϵ, δ, τ and a positive integer n such that

$$|\psi(\hat{x}\tau)| < \frac{\delta}{(|\hat{x}|\tau)^n} \tag{3.6}$$

for all $\tau > \tau_0$ and $\hat{x}$ in the complement of the domain

$$V_\epsilon(\psi) = (\hat{x} | |\hat{x}| \leqslant \epsilon; \text{ or } |\hat{x}t - \kappa| \leqslant \epsilon \text{ for some } \kappa \in \operatorname{supp} \psi;\ t \text{ real} \neq 0) \tag{3.7}$$

where $|x|$ is the Euclidean distance. In this way we can say that the fields $\psi(x)$ are confined to the region

$$V_\epsilon(\psi; \tau) = (x | x = \hat{x}\tau;\ \hat{x} \in V_\epsilon(\psi);\ \tau \text{ real}) \tag{3.8}$$

(II) *Causal Way of Propagation*

We assume that if a field $\psi(x)$, together with its first time derivative $\dot{\psi}(x)$ is confined to a finite region of M at a given time, then the velocity of propagation of its front is bounded by the velocity of light, in the meaning of Section 2.

We call a field theory defined by assumptions (a), (b) and (c) a *causal field theory* when causality requirements (I) and (II) are satisfied.

In our causal field theory we now introduce the following *causal relation*. We define a field at a point $x \in M$ to be causally related to a field at a point $x' \in M$, and we write

$$\psi(x)\,\mathscr{C}\,\psi(x') \tag{3.9}$$

if the following conditions hold:

(i) the fields $\psi(x)$ and $\psi(x')$ satisfy causality conditions (I) and (II), and can be interrelated by a propagation equation of type (3.3);

(ii) the separation is time-like

$$Q(x, x') > 0 \tag{3.10}$$

(iii) there is an ordering in time, e.g.

$$x^0 < x'^0 \tag{3.11}$$

The above causal relation introduces supplementary restrictions in our field theory. Indeed, it introduces an ordering in time and restricts the evaluation of the fields at any points x and x' time-like separated.

These restrictions are compatible with causality requirements (I) and (II). Indeed, relation $\mathscr{C}$ is compatible with condition (I), since it restricts only the intersection of the domains of localizability of $\psi(x)$ and of $\psi(x')$

$$T(x, x') = (V_\epsilon(\psi; \tau) \cap V_\epsilon(\psi', \tau') | Q(x, x') > 0;\, x^0 < x'^0;\, x \in V_\epsilon(\psi; \tau);\, x' \in V_\epsilon(\psi', \tau')) \tag{3.12}$$

to time-like separations and to the time ordering $x^0 < x'^0$, without acting in each single domain. Similarly, relation $\mathscr{C}$ is compatible with condition (II), since the causal way of propagation must occur everywhere in Minkowski space.

Finally, we introduce the concept of *causality group* of our field theory. The group of automorphisms G of the Minkowski space M is said to preserve causality, and is called a causality group, if any unitary representation T_g of G, for any $g \in G$, preserves relation $\mathscr{C}$; that is if a field $\psi(x)$ is causally related to a field $\psi(x')$, the same occurs for $T_g\psi(x)$ and $T_g\psi(x')$. On account of condition (i) of the causal relation, the above requirement implies that under any mapping

$$x \to x' = gx, \qquad g \in G \tag{3.13}$$

the fields remain causal. It also follows that, because of transformation property (3.2) and the composition law of group elements, any product of unitary representations $T_g T_{g'} T_{g''} \ldots$, with $g, g', g'' \ldots \in G$, preserves relation $\mathscr{C}$.

4. *Derivation of Poincaré Covariance from Causality Requirements*

The crucial consequence of the above causality requirements and causal relation can be expressed by the following theorem.

Theorem

The largest covariance group G of scalar fields $\psi(x)$, defined by assumptions (a), (b) and (c) with arguments x on a Minkowski space M, is uniquely restricted to the causality group C, if the preservation of causal relation $\mathscr{C}$ is assumed.

We prove this theorem by introducing first some lemmas.

Lemma 1

The non-linear automorphisms of the Minkowski space M which preserve causality condition (II) violate causality condition (I).

Proof: The causal way of propagation, condition (II), performs a first strong restriction on the covariance group G. Indeed, for the maximal case of propagation at the velocity of light, the largest group of automorphisms of a Minkowski space preserving a light signal is the conformal group (Cunningham, 1910; Bateman, 1910).

Consider now among the elements of the conformal group, the non-linear transformations given by the so-called accelerations

$$x \to x' = gx = \frac{x + bx^2}{1 + 2b_\alpha x^\alpha + b^2 x^2} \tag{4.1}$$

with

$$0 < b < \infty; \qquad b^2 = b^\mu b_\mu \neq 0; \qquad x^2 = x^\mu x_\mu \tag{4.2}$$

These transformations violate the localizability conditions of the fields, i.e., causality requirement (I). Indeed, by applying the mapping (4.1) to condition (3.4) we have

$$\lim_{\tau = \infty} \tau^n \psi[g(\hat{x}\tau)] = \psi\left(\frac{b}{b^2}\right) \lim_{\tau \to \infty} \tau^n \neq 0 \tag{4.3}$$

Equivalently, by recalling that the right-hand side of (3.6) satisfies the relation

$$\lim_{\tau = \infty} \frac{\delta}{(|\hat{x}|\tau)^n} = \lim_{\tau \to \infty} \frac{\delta}{|\hat{x}\tau|^n} = 0 \tag{4.4}$$

we have under mapping (4.1)

$$\lim_{\tau \to \infty} \frac{\delta}{|g(\hat{x}\tau)|^n} = \lim_{\tau \to \infty} \frac{\delta}{\left|\dfrac{\hat{x}\tau + b\hat{x}^2\tau^2}{1 + 2b_\alpha \hat{x}^\alpha \tau + b^2 \hat{x}^2 \tau^2}\right|^n} = \delta \left|\frac{b^2}{b}\right|^n \neq 0 \tag{4.5}$$

by which the violation of (3.6) follows by recalling that this condition must be satisfied for all $\tau > \tau_0$.

We can thus say that although the causal way of propagation might admit some classes of non-linear automorphisms, these transformations are forbidden by the localizability condition of the fields.

Let us now introduce the following *supplementary relations*. We define a field at a point $x \in M$ to be light-like related to a field at a point $x' \in M$, and we write

$$\psi(x)\,\mathscr{L}\,\psi(x') \tag{4.6}$$

if:

(i) fields $\psi(x)$ and $\psi(x')$ satisfy causality conditions (I) and (II), and can be interrelated by a propagation equation of type (3.3);

(ii) the separation is light-like

$$Q(x, x') = 0 \tag{4.7}$$

(iii) there is the ordering in time

$$x^0 < x'^0 \tag{4.8}$$

Finally we introduce relation $\tilde{\mathscr{C}}$ complementary to $\mathscr{C}$

$$\psi(x)\,\tilde{\mathscr{C}}\,\psi(x') \tag{4.9}$$

if

$$Q(x,x') \leqslant 0; \qquad x^0 < x'^0 \tag{4.10}$$

and relation $\tilde{\mathscr{L}}$ complementary to $\mathscr{L}$

$$\psi(x)\,\tilde{\mathscr{L}}\,\psi(x') \tag{4.11}$$

if

$$Q(x, x') \neq 0, \qquad x^0 < x'^0 \tag{4.12}$$

where, always, fields $\psi(x)$ and $\psi(x')$ satisfy causality conditions (I) and (II) and can be interrelated by equations of type (3.3).

Lemma 2

The group of automorphisms G of M preserves relation $\mathscr{C}$ if and only if it preserves relation $\mathscr{L}$.

Proof: We first note that if G preserves relation $\mathscr{C}$ and $\mathscr{L}$, then it preserves relation $\tilde{\mathscr{C}}$ and $\tilde{\mathscr{L}}$ too. Indeed, let us assume that $\psi(x)\,\tilde{\mathscr{C}}\,\psi(x')$, and that there exists a unitary representation T_g of G which produces the transition from the above relation to the new one $T_g\psi(x)\,\mathscr{C}\,T_g\psi(x')$. But then there exist a unitary representation T_{g-1} of G such that $T_{g-1}\psi(gx) = \psi(x)$ and $T_{g-1}\psi(gx') = \psi(x')$. Then $\psi(x)\,\mathscr{C}\,\psi(x')$ in contrast with the assumption. A similar situation occurs for relations $\mathscr{L}$ and $\tilde{\mathscr{L}}$.

The lemma then follows from the complementarity of relations $\mathscr{C}$, $\tilde{\mathscr{C}}$, $\mathscr{L}$, $\tilde{\mathscr{L}}$. Indeed, as for the corresponding geometrical case (Zeeman, 1964), relation $\psi(x)\,\mathscr{C}\,\psi(x')$ implies

$$\psi(x)\,\tilde{\mathscr{L}}\psi(x')$$
$$\psi(x)\,\mathscr{L}\psi(x'') \tag{4.13}$$

for some $\psi(x'')$ such that $Q(x,x'') = Q(x'',x') = 0$ and $x^0 < x''^0 < x'^0$. Thus, if G preserves relation $\mathscr{C}$, then it preserves relation $\mathscr{L}$ and $\tilde{\mathscr{L}}$.

Similarly, relation $\psi(x)\,\mathscr{L}\psi(x')$ implies

$$\psi(x)\tilde{\mathscr{C}}\psi(x')$$
$$\psi(x)\mathscr{C}\psi(x'') \tag{4.14}$$

for some $\psi(x'')$ such that $Q(x,x'') > 0$, $Q(x',x'') > 0$ $x^0 < x''^0$ and $x'^0 < x''^0$. Thus, if G preserves relation $\mathscr{L}$, then it preserves relation $\mathscr{C}$ and $\tilde{\mathscr{C}}$. Consequently, G preserves relation $\mathscr{C}$ if and only if it preserves relation $\mathscr{L}$.

The largest class of linear automorphisms of M preserving causal relations is now specified by the following lemma.

Lemma 3

The largest group of linear automorphisms G of the Minkowski space M preserving relation $\mathscr{L}$ is, modulo dilatations and preservation of time orderings, the group of all the isometries $I(M)$ of M.

Proof: Let us consider the domain of localizability of a couple of fields $\psi(x)$ and $\psi(x')$

$$V_\epsilon(\psi;\tau) = (x|x = \hat{x}\tau;\, \hat{x}\in V_\epsilon(\psi);\, \tau \text{ real})$$
$$V_{\epsilon'}(\psi;\tau') = (x|x' = \hat{x}'\tau';\, \hat{x}'\in V_{\epsilon'}(\psi);\, \tau' \text{ real}) \tag{4.15}$$

The introduction of the restrictive relation

$$\psi(x)\,\mathscr{L}\psi(x') \tag{4.16}$$

induces a metric-dependent domain expressed by the light-cone

$$0(x,x') = (V_\epsilon(\psi,\tau) \cap V_{\epsilon'}(\psi;\tau')|\,Q(x,x') = 0$$
$$x\in V_\epsilon(\psi;\tau);\, x'\in V_{\epsilon'}(\psi;\tau') \tag{4.17}$$

Now, corresponding to any unitary representation T_g of G preserving relation $\mathscr{L}$, i.e., such that

$$T_g\psi(x)\,\mathscr{L}T_g\psi(x') \tag{4.18}$$

the elements $g\in G$ must transform the light-cone $0(x,x')$ into another light-cone $0(x_1,x_1')$ with $x_1 = gx$ and $x_1' = gx'$. This implies that

$$Q(gx,gx') = Q(x,x') \tag{4.19}$$

Lemma 3 then follows by recalling that, by definition (Helgason, 1959), the group of all linear transformations on M preserving the metric tensor Q is the group of all the isometries $I(M)$ of M.

Proof of the Theorem

From Lemma 1 it follows that the group of automorphisms G of the Minkowski space M, in its most general structure is constituted by the group D of (positive) dilatation and by a (normal) subgroup A of linear transformations. From Lemmas 2 and 3 it follows that A is isomorphic, modulo the preservation of time ordering, to the group of all the isometries $I(M)$ of M. But $I(M)$ is given by the full Poincaré group (Helgason, 1959)

$$I(M) = P = L \times) T \tag{4.20}$$

where L is the full homogeneous Lorentz group and $\times)$ is the semidirect product. Then, the condition of time ordering restricts L to the orthochronous Lorentz group $L\uparrow$, and G becomes the causality group C.

5. *Concluding Remarks*

Let us recall that the Minkowski space M is a symmetric homogeneous admitting only one second-order differential operator invariant under the group of all isometries, given by the Laplace–Beltrami operator

$$\Delta = \frac{1}{\sqrt{g}} \partial^\mu g_{\mu\nu} \sqrt{g}\, \partial^\nu; \qquad \mu, \nu = 0, 1, 2, 3 \tag{5.1}$$

where $g = \det(g_{\mu\nu})$ and $g_{\mu\nu} = Q(\partial_\mu, \partial_\nu)$, while the algebra of all invariant differential operators in M consists of polynomials in Δ.

Operator (5.1) coincides in our case with the usual Laplacian

$$\Delta \equiv \Box = \partial_\mu g_{\mu\nu} \partial_\nu \tag{5.2}$$

on account of the fact that $g_{\mu\nu}$ is independent of x, the space M being flat.

We can thus say that the defining equation (3.1) of our theory of scalar fields uniquely reduces to equation (2.1) [particularly on account of the assumption on the order of (3.1)], when the covariance group of the theory is restricted to the preservation of causal relations. In this case the link between the group of automorphisms G of M and the Hilbert space H_ψ of our causal field theory is given by the Laplace–Beltrami operator of the space and the full theory of Poincaré covariant scalar fields is recovered.

For the above derivation of Poincaré covariance from causality requirements some assumptions less restrictive than (a), (b) and (c) of Section 3 could also be considered. We quote, for instance:

(1) the considered class of solutions of (3.1) could be enlarged;
(2) the condition that (3.1) is a second-order differential equation could be generalized;
(3) the condition that the function ψ is preserved after propagation (3.3) could be relaxed;
(4) the propagation equation (3.3) could be generalized by introducing hypersurfaces in M;
(5) the fields could be introduced in terms of transformation laws more general then (3.2), such as, for instance,

$$T_g \Psi(x) = A(x; g) \Psi(gx), \; g \in G \tag{5.3}$$

where (x) is a column vector with components $\psi_\alpha(x)$, and $A(x,g)$ is a non-singular operator satisfying the composition law

$$A(x;y), A(x;y_2) = A(x;g_1 g_2); g_1, g_2 \in G \tag{5.4}$$

in order to recover theories of vectorial and spinorial fields.

Conceivably, the above generalizations should not essentially affect the proof of the theorem, the only different result being constituted by the transition from the orthochronous Poincaré group to its universal covering group when spinorial fields are recovered.

On the contrary, the condition on the linearity of equation (3.1) seems to be essential on account of the small number of information that we have at the present time on causality in a non-linear theory.

It is also interesting to note that the above derivation of Poincaré covariance from causality requirements in field theories is independent of whether $\psi(x)$ is a classical field or a field in first quantization. This implies that causality requirements (I) and (II) and causal relations, which apply in classical and first quantum-mechanical field theories, lead to the derivation of Poincaré covariance in both cases.

The extension of the above result to field theories in second quantization requires, however, specific supplementary investigations. Indeed, the same basic assumptions must be modified on account of the fact that the fields now become operator valued distributions. This opens the interesting problem of whether, and which, causality requirements in second quantized field theories are able to recover Poincaré covariance.

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References

Bateman, H. (1910). *Proceedings of the London Mathematical Society*, **8**, 223 and 454.
Chandler, C. (1968). *Physical Review*, **174**, 1749.
Chandler, C. (1969). *Physical Review*, **179**, 1632 (E).
Cunningham, E. (1910). *Proceedings of the London Mathematical Society*, **8**, 77.
Helgason, S. (1959). *Acta mathematica*, **102**, 239.
Hilgewoord, J. (1962). *Dispersion Relations and Causal Description*, North-Holland Publishing Co., Amsterdam.
Zeeman, E. C. (1964). *Journal of Mathematics and Physics*, **5**, 490.